SOME CONTRACTIBLE OPEN 3-MANIFOLDS

BY

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1. **Introduction.** The purpose of this paper is to present 3 basically different types of contractible open 3-manifolds. In §3, an uncountable collection of topologically different 3-manifolds of the first type is constructed. According to [4], each of these spaces yields E^4 when multiplied by a line. In §4, a contractible open 3-manifold is constructed which is not the union of a properly ascending sequence of solid tori (i.e., cubes with 1 handle). In §5, an example of Bing is considered(1) and some results are obtained concerning the possibility of embedding this 3-manifold in S^3.

These examples provide answers to several questions about such 3-manifolds and emphasize their complexity. J. H. C. Whitehead [9] gave the first example of such a space which is different from E^3. Theorem 1 of [4] asserts that a generalization of Whitehead's construction is essentially the only way to construct such spaces. No attempt is made to make these constructions independent of the figures employed or of the choice of particular mappings.

2. **Definitions.** An *n-manifold* is a countable locally-finite, connected simplicial complex such that the link of each vertex is piecewise linearly homeomorphic to the standard $(n-1)$-sphere. An *open manifold* is without boundary and non compact; a *closed manifold* is without boundary and compact. All spaces and mappings are taken in the polyhedral or piecewise linear sense, and all of the manifolds considered are orientable.

Let T be a solid torus, and J a tame simple closed curve in Int T. Suppose that D is a polyhedral disk in T such that $D \cdot \text{Bd}\ T = \text{Bd}\ D$, Bd D does not separate Bd T, and J pierces D at each point of $J \cdot D$ (the first two requirements imply that the manifold obtained by cutting T along D is a 3-cell). Denote by $N(J, T)$ the minimum of $J \cdot D$ for all disks with the above properties. Note that $N(J, T)$ is a non-negative integer which is 0 if and only if a polyhedral cube in Int T contains J. If $N(J, T) = k$, it will also be said that *J wraps around T k times*. This concept was introduced by Schubert [8]. It is easy to verify that $N(J, T)$ is a topological invariant of the pair (J, T).

If T_0 and T_1 are polyhedral solid tori with $T_0 \subseteq T_1$, one denotes by $N(T_0, T_1)$ the integer $N(J, T_1)$, where J is any polyhedral simple closed curve in Bd T_0 which circles T_0 exactly once longitudinally. Since any 2 such J's are equivalent under a homeomorphism of T_1, $N(T_0, T_1)$ is well-defined. The number of times that a simple closed curve wraps around a solid torus will,

Received by the editors May 19, 1960.

(1) Some properties of this example were presented to the Society, November 19, 1960 under the title *A certain contractible open 3-manifold.*

in general, differ from the number of times that the simple closed curve "circles" the torus (see [3]). The longitudinal simple closed curve of T_0 in Figure 5, for example, circles the torus 0 times, but wraps around it 2 times. Note, however, that the absolute value of the "circling number" never exceeds $N(J, T)$, and in certain simple cases equality holds.

Schubert [8] proved the following:

LEMMA 1. *Let T_0, T_1, T_2, be polyhedral solid tori with*

$$T_0 \subseteq \text{Int } T_1 \subseteq T_1 \subseteq \text{Int } T_2.$$

Then

$$N(T_0, T_2) = N(T_0, T_1) \cdot N(T_1, T_2).$$

This lemma will be used in later proofs and is useful in verifying certain assertions about wrapping numbers.

3. **Uncountably many divisors of E^4.** Let the letter p denote an infinite sequence of distinct primes each greater than $2: p_1, p_2, p_3, \cdots$. Corresponding to p, construct a space W_p as described in the next paragraph. Note that there exists a collection with cardinality of the continuum, each element of which is a sequence of the above type and such that any two such sequences have only a finite number of primes in common. Theorem 2 follows from this remark and Theorem 1.

Let T_1^n $(n \geqq 0)$ be a countable collection of mutually exclusive unknotted polyhedral solid tori in E^3. In the interior of T_1^n, choose an unknotted polyhedral solid torus T_0^n such that each simple closed curve in T_0^n can be shrunk to a point in T_1^n and such that T_0^n wraps around T_1^n precisely $2p_{n+1}$ times $(n \geqq 0)$. Figure 1 illustrates how this can be done when $p_{n+1} = 3$, and it is clear that it can be done in general. Define homeomorphisms $h_0, h_1, h_2, \cdots$, of E^3 onto itself as follows. The mapping h_0 is the identity; h_1 throws T_0^1 onto T_1^0; h_2 throws T_0^2 onto $h_1(T_1^1)$; h_3 throws T_0^3 onto $h_2(T_1^2)$, and so on.

Define:

$$W_p = T_1^0 + h_1(T_1^1) + h_2(T_1^2) + h_3(T_1^3) + \cdots.$$

Setting $H_i = h_{i-1}(T_1^{i-1})$ for $i \geqq 1$, notice the following:

(1) $W_p = \sum_{i=1}^{\infty} H_i$, where H_i is an unknotted solid torus and $H_i \subseteq \text{Int } H_{i+1}$;

(2) each loop in H_i can be shrunk to a point in H_{i+1};

(3) H_i wraps around H_{i+1} exactly $2p_{i+1}$ times.

The first 2 of the above properties imply that W_p is an open subset of E^3 with trivial homotopy groups and, hence, is contractible by [10].

THEOREM 1. *Suppose p and q are sequences of primes (as described above) such that an infinite number of primes occur in p which do not occur in q. Then, W_p and W_q are topologically different.*

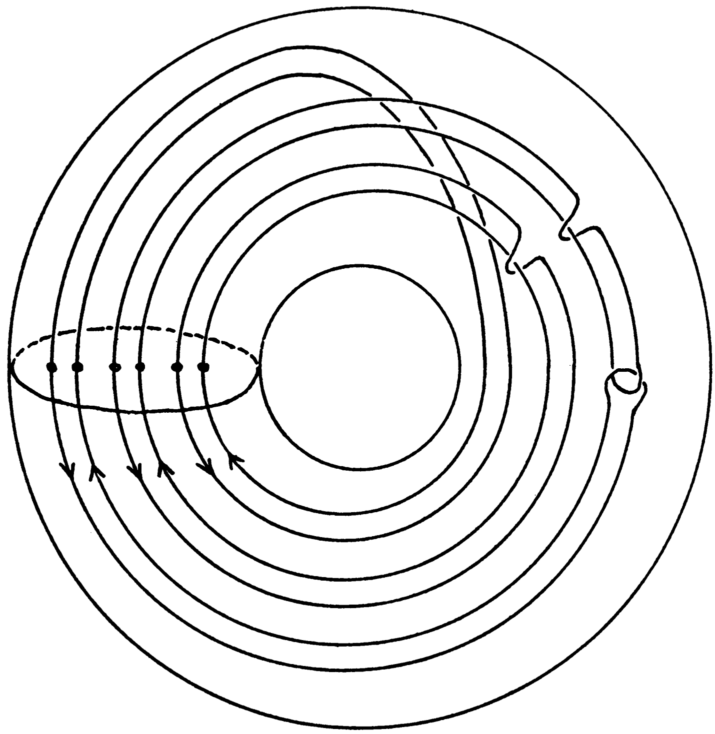

FIGURE 1

Proof. Let $W_p = \sum_{i=1}^{\infty} H_i$ and $W_q = \sum_{i=1}^{\infty} R_i$ where the solid tori H_i and R_i satisfy the appropriate conditions for the sequences p and q respectively (see the above three properties).

First, if J is a longitudinal simple closed curve in the boundary of one of these solid tori, say $J \subseteq \text{Bd } H_1$, then the interior of no polyhedral 3-cell in W_p contains J. If this were not so, then for some integer n, $N(J, H_n) = 0$, in contradiction to Lemma 1 and property 3.

Now suppose h is a homeomorphism of W_p onto W_q. By [5, Theorem 2], h may be assumed to be locally piecewise linear. Select positive integers j, k, and m so that $R_1 \subseteq \text{Int } h(H_j)$, p_k occurs in the sequence p but not in q $(j+1 < k)$, and $h(H_k) \subseteq \text{Int } R_m$. By Lemma 1,

$$N(R_1, R_m) = N(R_1, h(H_j)) \cdot N(h(H_j), h(H_k)) \cdot N(h(H_k), R_m).$$

This equation must be incorrect, however, since none of these integers is 0, while p_k divides $N(h(H_j), h(H_k))$ but does not divide $N(R_1, R_m)$. This contradiction establishes Theorem 1.

THEOREM 2. *There exist uncountably many contractible open subsets of* E^3, *no two of which are homeomorphic. Hence, there are uncountably many different ways to express* E^4 *as the product of a 3-manifold and a line.*

4. The second example. In [3], the author showed that a closed 3-manifold M is topologically S^3 if each of its simple closed curves can be shrunk to a point inside a solid torus in M. The question naturally arises as to whether simple-connectivity alone is enough to imply this condition. This example shows the answer to be in the negative in the absence of compactness.

The following was shown in [3, Theorem 1].

LEMMA 2. *Let M be a 3-manifold without boundary such that each simple closed curve in M can be shrunk to a point in a solid torus in M. If G is a polyhedral, connected finite graph in M each of whose points is of order 2 or 4, then G is contained in the interior of a polyhedral solid torus in M.*

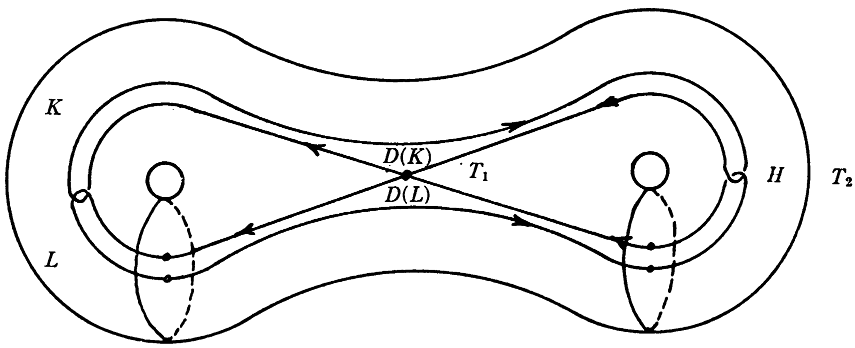

FIGURE 2

Consider Figure 2. This represents a double torus T_1 (cube with 2 handles) in the interior of another double torus T_2, although T_1 is drawn as though it were an 8-curve (topological figure 8). These tori are embedded as shown as polyhedral subsets of E^3.

There is a homeomorphism h of E^3 onto itself which throws T_1 onto T_2 and which is the identity in the exterior of some sphere containing T_2. Define U to be the sum of the following properly ascending sequence of sets: T_1, $h(T_1)$, $h^2(T_1)$, $\cdots$, $h^n(T_1)$, $\cdots$. It is easily verified that U is a contractible open subset of E^3. Denote by H the 8-curve in Int T_2 which when expanded slightly gives T_1.

LEMMA 3. *The 8-curve H is not contained in the interior of any solid torus in U.*

Proof. Suppose that T is any punctured torus (a solid torus from whose interior have been removed the interiors of a finite number of mutually ex-

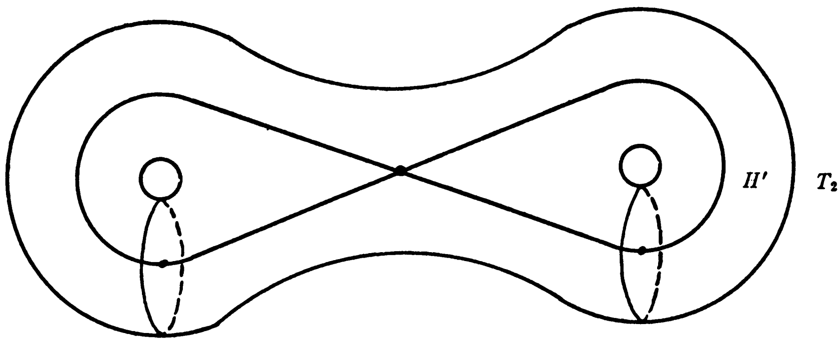

FIGURE 3

clusive tame 3-cells) in E^3 whose interior contains H. It will be shown that [Int T]·[Int T_2] contains an 8-curve H' embedded in T_2 as shown in Figure 3, in the sense that there is an isotopy of T_2 onto itself, fixed on Bd T_2 and taking H' into the position indicated. Now apply the same argument to H' and $h(T_2)$ to obtain a curve H'', and so on. Hence, it will follow that, for each n, [Int T]·[Int $h^n(T_1)$] contains such an 8-curve, and hence, for each n, T contains a simple closed curve which cannot be shrunk to a point in $h^n(T_1)$. Thus T cannot lie in U. There is no loss in supposing that T is polyhedral.

The 8-curve H is the sum of the simple closed curves K and L which bound polyhedral 2-cells $D(K)$ and $D(L)$, respectively, in Int T_2 that intersect only in disjoint arcs α and β such that $\text{Int}\,\alpha + \text{Int}\,\beta = [\text{Int}\,D(K)] \cdot [\text{Int}\,D(L)]$. It may be supposed that Int $D(K)$ and Bd T are in general position, and that Int $D(L)$ and Bd T are in general position.

A simple closed curve in one of the sets [Bd T]·[Int $D(K)$] or [Bd T]·[Int $D(L)$] will be said to be *negligible* if it bounds a disk in Int $D(K)$ or Int $D(L)$ which misses H. It will be convenient to simplify further the relation of Bd T to $D(K)$ and $D(L)$ by assuming in what follows that neither of these disks contains a negligible curve.

This assumption is justified as follows: There is a sequence of operations beginning with T and leading to a punctured torus (or punctured cube) whose boundary is in general position relative to Int $D(K)$ and Int $D(L)$ and whose boundary contains no negligible curves. Each of these operations is applied to the punctured torus obtained from the previous operation and consists of cutting along a disk or attaching a 3-cell along an annulus ring. Each operation eliminates an "inner" negligible curve. For more details, see the proof of Lemma 7 of [3].

The proof given below shows that the required 8-curve H' can be found in Int T'. Then, there is an isotopy of T_2 which is fixed on Bd T_2 and which pushes H' into Int T. Hence, the deformed H' will still lie in Int T_2 in the desired manner. The required isotopy is the product of a finite sequence of isotopies of T_2, each of which is the identity outside a small neighborhood of a disk in Int $D(K)$ or Int $D(L)$ whose boundary is a negligible curve.

Under the hypothesis that Bd T contains no negligible curves, it will be shown that at least one of $D(K)$ and $D(L)$ lies in Int T, from which the existence of H' follows immediately. If (say) $D(K)$ is not contained in Int T, then there is a polygonal simple closed curve J in Bd T bounding a polyhedral disk $D(J)$ in Int $D(K)$ such that $L \cdot \text{Int}\, D(J) = H \cdot \text{Int}\, D(J)$ consists of i points ($i=1$ or 2) and Int $D(J) \subseteq$ Int T. Now, $D(J)$ does not separate T. This is clear in case $i=1$, and in case $i=2$, note that L pierces Int $D(J)$ from the same side at the points of intersection.

Hence, a closed curve in T can be shrunk to a point in T if and only if its linking number (with integer coefficients) with respect to Int $D(J)$ is 0. Thus, K can be shrunk to a point in Int T and cannot link (again, with integer coefficients) any simple closed curve on Bd T. It follows that Bd T misses Int $D(L)$ and Lemma 3 follows.

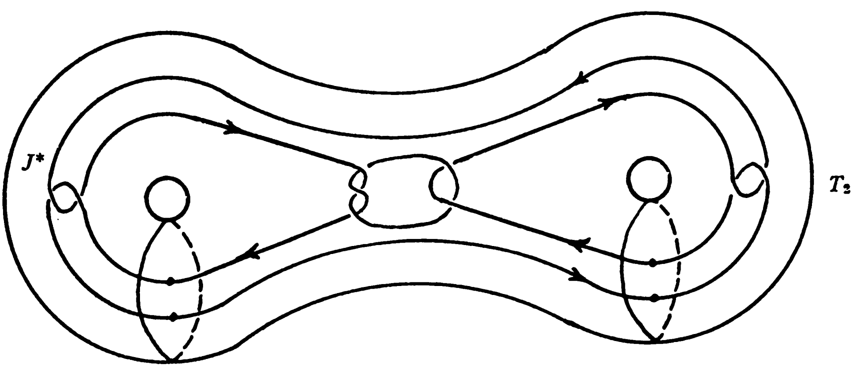

FIGURE 4

THEOREM 3. *The simple closed curve J^* indicated in Figure 4 lies trivially in no solid torus in U.*

Proof. This follows from Lemma 3 and the proof of Lemma 2, as contained in [3, Theorem 1].

5. An example of Bing. If each compact subset of the contractible open 3-manifold U can be embedded in E^3, does it follow that U can be embedded in E^3? Theorem 2 of [4] shows that U can be embedded in E^4. R. H. Bing has conjectured that the example U^* described here will provide a negative answer to the above question. The author does not settle this question, but does show that, if U^* has an embedding in S^3, it must be an extremely "tangled" one. It is not hard to vary the construction given below to obtain a non-simply-connected open 3-manifold which is the union of an ascending sequence of solid tori, but which cannot be embedded in E^3.

Consider the 2 solid tori T_0, T_1, of Figure 5 in E^3, where T_0 lies in Int T_1 as indicated. Let X^0 be the set $T_1 - \text{Int } T_0$, so that X^0 is a compact 3-manifold with 2 boundary components, an "inner" one X^0_0 and an "outer" one X^0_1. Let f_n ($n=0, 1, 2, \cdots$) be translation homeomorphisms of E^3 such that f_0 is the identity and the images $f_n(T_1)$ are all mutually exclusive for $n \geqq 0$. Let $X^n = f_n(X^0)$, $X^n_0 = f_n(X^0_0)$, and $X^n_1 = f_n(X^0_1)$.

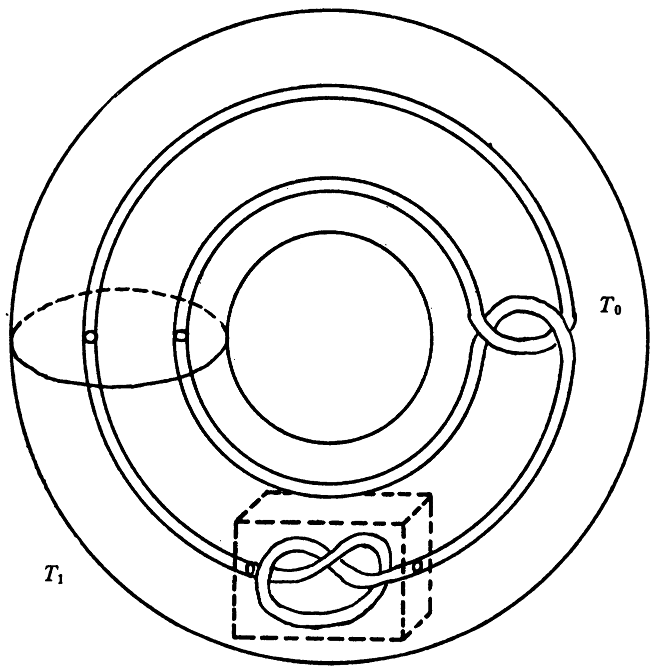

FIGURE 5

Consider also the homeomorphisms:

$$g_n: X^n_1 \to X^{n+1}_0 \qquad (n \geqq 0),$$

where the homeomorphism g_n takes a meridianal simple closed curve of the nth translate of T_1 onto a meridianal simple closed curve of the $(n+1)$st translate of T_0. A longitudinal simple closed curve will also be taken onto a longitudinal simple closed curve, and there are many different choices for g_n. An arbitrary selection is made here.

The space U^* is obtained from $T_0 + \sum_{n=0}^{\infty} X^n$ by identifying each x in X^n_1 with its image $g_n(x)$. Since $\{f_n(T_1) \mid n=0, 1, 2, \cdots\}$ is a discrete collection of closed sets, the topology on U^* may be characterized by declaring a subset of U^* to be closed if and only if its inverse image under the identification map is closed in E^3. Clearly, if H_i denotes the solid torus in U^* obtained by identifying certain points of $T_0+\sum_{n=0}^{i} X^n$, then $U^*=\sum_{i=0}^{\infty} H_i$, where $H_i \subseteq \text{Int } H_{i+1}$ and each loop in H_i can be shrunk to a point in H_{i+1}. U^* is a contractible open 3-manifold.

Recall some definitions. A 3-manifold X with nonempty boundary is *irreducible* if each loop in Bd X which can be shrunk to a point in X can also be shrunk to a point in Bd X. Otherwise, X is *reducible.*

LEMMA 4. *X is reducible if and only if there is a polyhedral disk D in X such that $D \cdot$* Bd $X=$ Bd D *and* Bd D *does not bound a disk in* Bd X.

This follows from the Loop Theorem [6] and Dehn's Lemma [7] of Papakyriakopoulos.

Suppose for the rest of this section that h is any homeomorphism of U^* into S^3. By [5, Theorem 2], h may be supposed to be locally piecewise linear, without altering the set $h(U^*)=M$. In the following statements involving fundamental groups, the question of base points is irrelevant, and will not be mentioned.

LEMMA 5. *If C is a compact subset of M, then the inclusion homomorphism*

$$i^*: \pi_1(M - C) \rightarrow \pi_1(S^3 - C)$$

has trivial kernel.

Proof. If C lies in the interior of a topological cube in M, then i^* is an isomorphism onto, and there is nothing to prove. Suppose C lies in the interior of no topological cube in M. Lemma 4 then implies that if T is a solid torus such that $C \subseteq \text{Int } T$, then the 3-manifold $T-C$ is irreducible. This will be used below.

First, $h(H_n)$ must be a knotted solid torus in S^3 (H_n was defined earlier). For there is a tame topological cube F (see Figure 5) in S^3 which lies in $h(H_{n+1})$ and whose intersection with $h(H_n)$ is a knotted 3-cell G. More precisely, there is a simple closed curve J in Int $F-G$ which is the boundary of a compact 2-manifold in this set, but which cannot be shrunk to a point here. But J cannot be shrunk to a point even in $S^3-h(H_n)$, since by Lemma 4 the manifolds $F-G$ and $[(S^3-\text{Int } F)-h(H_n)]$ are irreducible. Since J represents a nontrivial commutator element, $\pi_1(S^3-h(H_n))$ is nonabelian.

Thus, $h(H_n)$ is knotted and $S^3-\text{Int } h(H_n)$ is irreducible. If N is an integer so large that $C\subseteq \text{Int } h(H_N)$, then the 3-manifold $h(H_n)-C$ is irreducible for $n \geqq N$ and Lemma 5 follows.

A cube with handles R in S^3 is said to be *unknotted* if there is a homeomorphism of S^3 onto itself taking R onto a canonical cube with handles (see

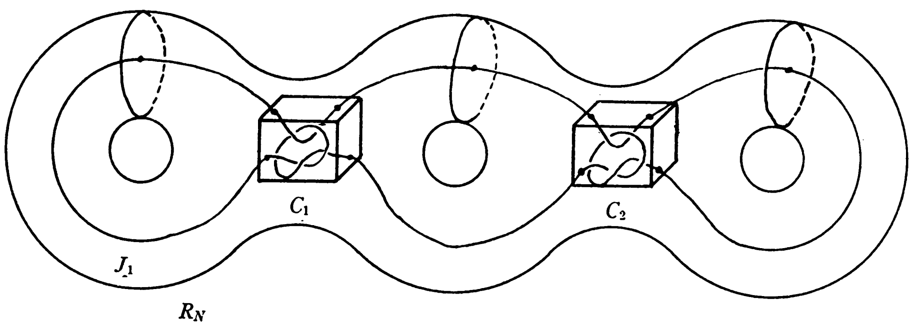

FIGURE 6

Figure 6). The following property distinguishes U^* from the other examples given in this paper.

THEOREM 4. *Suppose that* $M=\sum_{n=1}^{\infty} R_n$, *where* R_n *is a polyhedral cube with handles, and* $R_n \subseteq \text{Int } R_{n+1}$. *Then, only a finite number of the* R_n*'s are unknotted.*

Proof. If not, suppose each R_n is unknotted. Let J_0 be a longitudinal simple closed curve in Bd H_0. Then, the interior of no topological cube in M contains $h(J_0)$. This can be shown as in the proof of Theorem 1, using wrapping numbers. There also exists such a curve which is unknotted in S^3. For, let N be an integer for which $h(J_0) \subseteq \text{Int } R_N$ and suppose that R_N is nicely situated in S^3 (see Figure 6).

Let J_1 be the unknotted simple closed curve indicated in Figure 6. If a topological cube H in M contained J_1, there would be, as in [1, Lemma 7] a topological cube H' containing $N(J_1)+\sum C_i$, where $N(J_1)$ is a small tubular neighborhood of J_1 and the 3-cells C_i are as shown in Figure 6. There is a homeomorphism of M onto itself which is the identity outside a small neighborhood of R_N and which takes $N(J_1)+\sum C_i$ onto R_N and H' onto a topological cube in M containing R_N, a contradiction. Thus, J_1 is the required simple closed curve.

Since $\pi_1(S^3-J_1)$ is abelian, $\pi_1(M-J_1)$ is abelian, by Lemma 5. It is then an easy consequence of Dehn's Lemma [7] that J_1 bounds a 2-cell in M and hence lies in a 3-cell in M. This contradiction establishes the theorem.

The following shows that M could not, for example, be the complement of a wild arc in S^3. An example of an arc is given in [2] with the property that its complement is simply-connected, but not topologically E^3.

THEOREM 5. *The dimension of* B, *the boundary of* M *in* S^3, *is* 2.

Proof. Suppose that dim $B \leqq 1$. Consider the knotted solid torus $h(H_n)$, for a fixed n. There are mutually exclusive 3-cells $\{C_i\}$ in S^3 such that

$h(H_n)\cdot C_i=[\text{Bd } C_i]\cdot[\text{Bd } h(H_n)]$ consists of 2 mutually exclusive 2-cells E_i and F_i, C_i does not separate M and $h(H_n)+\sum C_i$ is an unknotted cube with handles. The C_i's are obtained by considering a projection of the knot associated with $h(H_n)$.

Since B does not separate Int C_i, there is an arc α_i (possibly knotted) from a point p_i of E_i to a point q_i of F_i such that Int $\alpha_i \subseteq \text{Int } C_i - B$. Hence, $\alpha_i \subseteq M$. Let β_i be an arc from p_i to q_i in M whose interior misses C_i. Then $\alpha_i+\beta_i$ is a simple closed curve in M which links any simple closed curve in Bd C_i that separates E_i from F_i. Hence, if f is a mapping of a disk D into M that shrinks $\alpha_i+\beta_i$ to a point, some component of

$$f(D)\cdot[\text{Bd } C_i - (\text{Int } E_i + \text{Int } F_i)]$$

meets both E_i and F_i. Thus, there is an arc γ_i in

$$M\cdot[\text{Bd } C_i - (\text{Int } E_i + \text{Int } F_i)]$$

from E_i to F_i.

The existence of the arcs γ_i above permits the 3-cells C_i to be replaced by mutually exclusive tubular neighborhoods in M of the γ_i's. Hence, there is an unknotted cube with handles in M containing $h(H_n)$. This contradicts Theorem 4 and completes the proof.

QUESTION. Can U^* be embedded in S^3? Perhaps it could be shown that if the embedding h exists then the polyhedral simple closed curve J_0 (see Theorem 4) maps onto a simple closed curve $h(J_0)$ so wild that $\pi_1(S^3-h(J_0))$ is not finitely generated. Theorem 4 suggests another approach.

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